Lecture- 7

Laplace’ and Poisson’s Equations

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Derivation

We have the differential form of Gauss' law

$$\nabla . \vec{D} = \rho$$

Using **D** = ε **E** and **E =** - ∇V in the above equation, we get

$$\nabla \cdot \vec{\mathrm{D}} = \nabla \cdot \varepsilon \vec{E} = \varepsilon \nabla \cdot \left(-\nabla \mathrm{V} \right) = -\varepsilon \nabla^2 \mathrm{V} = \rho$$

$$or$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$

This is the Poisson's Equation

In free space , this equation becomes,

$$\boxed{\nabla^2 V = 0}$$

This equation is called the Laplace' equation.

∇^2 is called the Laplacian operator or simply Laplacian.

Note that the 'del' operator ∇ is defined only in the rectangular coordinates only, as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

The Laplacian in the three coordinate systems are

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \mathbf{(}\mathit{cartesian}\mathbf{)}$$

$$\nabla^2 = \frac{\mathbf{1}}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \mathbf{(}\mathit{Cylindrical}\mathbf{)}$$

$$\nabla^2 = \frac{\mathbf{1}}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \mathbf{sin}\,\theta}\frac{\partial}{\partial \theta}\left(\mathbf{sin}\,\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \mathbf{sin}^2\theta}\frac{\partial^2}{\partial \phi^2} \mathbf{(}\mathit{Spherical}\mathbf{)}$$

Applications

We look for one dimensional solution of Laplace' equation in each of the three coordinate systems, starting first from the rectangular coordinate system. That is,

V is a function of only one variable and is independent of the other two variables. Under this condition, in rectangular coordinate system the Laplace' equation reduces to

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \mathbf{0} \qquad \mathbf{(}cartesian\mathbf{)} \quad \text{---} \quad \mathbf{(1)}$$

One dimensional solution of Laplace' Equation in rectangular coordinate system

Let V be a function of z only. Then in Rectangular coordinate system,the Laplace's Equation reduces to

$$\nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{--- (2)}$$

Since V is a function of z only, it is independent of x and y. Therefore,

Integrating both sides once we get

$$\frac{dV}{dz} = A,$$ A is an arbitrary constant --- **(3)**

Integrating both sides once again, we get

$$V = Az + B,$$ B is an arbitrary constant --- **(4)**

A and B are arbitrary constants to be evaluated under suitable boundary conditions.

Equation (3) represents <u>a family of equi – potential surfaces</u> with z taking up constant values

Consider two such equi – potential surfaces one at $z = z_1$ and the other at $z = z_2$. Let $V = V_1$ at $z = z_1$ and $V = V_2$ at $z = z_2$.

We immediately recognize that this is the case with a parallel plate capacitor with a plate separation of $z_1 - z_2 = d$ and a potential difference $V_1 - V_2$.

Applying the above two conditions, called boundary conditions, we get,

$$V = V_1 = Az_1 + B \quad \text{--- (5)}$$

$$V = V_2 = Az_2 + B \quad \text{--- (6)}$$

Solving equations (5) and (6) we get the values for A and B as

$$A = \frac{V_2 - V_1}{z_2 - z_1}$$

$$B = \frac{V_1 z_2 - V_2 z_1}{z_2 - z_1}$$

Substituting the values of A and B in equation (4) we get,

$$V = \frac{V_2 - V_1}{z_2 - z_1} z + \frac{V_1 z_2 - V_2 z_1}{z_2 - z_1}$$

$$= \frac{V_2(z - z_1) - V_1(z - z_2)}{z_2 - z_1} \quad \text{--- (7)}$$

Further let, for simplicity, $V_1 = 0$ and $z_1 = 0$, $V_2 = V_a$

Fig 1 Parallel plate capacitor

Then equation (7) reduces to

$$V = \frac{V_a}{d} z \qquad \text{--- (8)}$$

We find that V is a linear function of z

Since we have an expression for the potential at any point between the plates of a parallel plate capacitor, we can make use of it to determine the capacitance of the parallel plate capacitor by following the steps:

1 Given V, Determine **E** using the formula **E** = - ∇ V

2 Determine **D** using **D** = ε **E**

3 Find D on any one of the plates, **D** = $\mathbf{D}_s$= D_S $\mathbf{a}_S$ = D_N $\mathbf{a}_N$ on the chosen plate, and recognising that $D_N = \rho_S$

4 Determine Q by surface integration of ρ_S over the surface area of the chosen plate using $Q = \int_S \rho_S dS$

5 Compute the capacitance using the formula $C = \frac{|Q|}{V_a}$

Applying these five steps to the parallel plate capacitor we get,

$$V = V_a \frac{z}{d}$$

$$\vec{E} = -\nabla V = \frac{V_a}{d} \hat{a}_z$$

$$\vec{D} = \varepsilon \vec{E} = -\varepsilon \frac{V_a}{d} \hat{a}_z$$

$$\vec{D}_S = \vec{D}\Big|_{z=0} = -\varepsilon \frac{V_a}{d} \hat{a}_z = D_S \hat{a}_S = D_N \hat{a}_N ; D_N = \rho_S$$

$$Q = \int_S -\varepsilon \frac{V_a}{d} dS = -\varepsilon \frac{V_a}{d} \int_S dS = -\varepsilon \frac{V_a}{d} S$$

Therefore the capacitance of the parallel plate capacitor is

$$C = \frac{|Q|}{V_a} = \frac{\varepsilon S}{d}$$